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Numerical calculation of decaying isotropic turbulence using the LET theory

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The local-energy-transfer (LET) theory (McComb 1978) was used to calculate freely decaying turbulence for four different initial spectra at low-to-moderate values of microscale Reynolds numbers (R_λ up to about 40). The results for energy, dissipation and energy-transfer spectra and for skewness factor all agreed quite closely with the predictions of the well-known direct-interaction approximation (DIA: Kraichnan 1964). However, LET gave higher values of energy transfer and of evolved skewness factor than DIA. This may be related to the fact that LET yields the $k^{-\frac{5}{3}}$ law for the energy spectrum at infinite Reynolds number.

The LET equations were then integrated numerically for decaying isotropic turbulence at high Reynolds number. Values were obtained for the wavenumber spectra of energy, dissipation rate and inertial-transfer rate, along with the associated integral parameters, at an evolved microscale Reynolds number R_λ of 533. The predictions of LET agreed well with experimental results and with the Lagrangian-history theories (Herring & Kraichnan 1979). In particular, the purely Eulerian LET theory was found to agree rather closely with the strain-based Lagrangian-history approximation; and further comparisons suggested that this agreement extended to low Reynolds numbers as well.

1. Introduction

As is well known, the fundamental problem in the theory of well-developed turbulence is the need to close the infinite hierarchy of moment equations. This hierarchy is generated when one averages the Navier–Stokes equations. Thus the 'closure problem', in one form or another, stands in the way of engineering calculations and fundamental studies alike. In the former, it is normally seen as the need to relate single-point quantities (e.g. the mean velocity, the Reynolds stress) to each other. In the latter, it involves two-point quantities and is usually studied in the context of isotropic turbulence, where the mean rate of shear is zero and the moment hierarchy may be studied in isolation. Although this introduces some artificialities, and indeed problems in making direct comparisons with experiment, these are generally taken to be outweighed by the reduced level of complexity. General discussions of the study of isotropic turbulence will be found in the books by Batchelor (1971) and Leslie (1973).

In this paper we are concerned with the numerical calculation of a particular closure approximation for isotropic turbulence (McComb 1974, 1976, 1978: the last one to be cited hereafter as I). This closure is based on the idea that the response of the system may be calculated from the local (in wavenumber) behaviour of the renormalized energy equation. Closure is achieved in terms of the energy spectrum

and the velocity-field propagator; and second-order equations have been obtained for these quantities (see I). We shall call it the local-energy-transfer (LET) theory.

As a second-order two-point closure, the LET theory belongs to a class which may be loosely described as renormalized perturbation theories (Kraichnan 1959, 1965; Wyld 1961; Edwards 1964; Herring 1965, 1966; Lee 1965; Edwards & McComb 1969; Phythian 1969; Balescu & Senatorski 1970; Nakano 1972). The general approach involves renormalizing a perturbation series for the energy spectrum, in terms of a second function (variously called the response function, velocity-field propagator or effective viscosity). Only second-order terms of the renormalized expansion are retained. Each theory leads to (essentially) the same equation for the energy spectrum, whereas differences in approach tend to be reflected in different equations for the response function.

A discussion of some of these theories will be found in Leslie (1973). Also, more-recent functional formulations have been given (Martin, Siggia & Rose 1973; Phythian 1975, 1976). Here we shall just make a few remarks in order to put the present work in perspective.

Turbulence research was much influenced during the 1960s by the Kolmogorov (1941) theory, which predicts that the energy spectrum in the inertial range is given by

$$E(k) = \alpha \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}, \tag{1.1}$$

where ϵ is the rate of energy dissipation, k is the wavenumber and α is a constant. This form is in good agreement with experiment (Grant, Stewart & Moilliet 1962, Champagne 1978), but higher-order moments are sensitive to intermittent effects and the Kolmogorov theory gives less satisfactory results for these. There is therefore a growing school of thought (see e.g. Kraichnan 1974; Frisch, Sulem & Nelkin 1978) that the inertial-range spectrum should take the form

$$E(k) = \alpha \langle \epsilon \rangle^{\frac{2}{3}} k^{-\frac{5}{3}} (kL_0)^{-\frac{1}{3}\mu}, \tag{1.2}$$

where $\langle \epsilon \rangle$ is the mean dissipation rate, L_0 is the lengthscale of the largest eddies and μ is the exponent in a power-law form for the spatial covariance of the fluctuating dissipation.

During the 1960s it seems to have been widely assumed that closure approximations should be compatible with (1.1). Indeed it was the failure of DIA (Kraichnan 1959), the functional probability method (Edwards 1964), and other similar methods, to yield (1.1) at infinite Reynolds number, which led to further attacks on the closure problem (Kraichnan 1965; Edwards & McComb 1969). In particular, Kraichnan (1965) reworked the direct-interaction approximation (DIA) in a Lagrangian-history coordinate system in order to ensure invariance under random Galilean transformations. Also, Edwards & McComb (1969) modified the functional formalism of Edwards (1964) to maximize the turbulent entropy. Both these theories have the $k^{-\frac{5}{3}}$ spectrum as the infinite-Reynolds-number solution, although the former requires a Lagrangian frame and the latter is independent of time. So far the main claim of LET (see I) is that it is a time-dependent theory which is compatible with the Kolmogorov distribution within an Eulerian coordinate system.

At a fundamental level, the relevance of this claim may seem unclear. If (1.2) is the correct solution (that is, if $\mu \neq 0$, although quite certainly μ is too small for measurements of $E(k)$ to distinguish between the two forms given above), can (1.1) still be regarded as a fundamental test for turbulent closures? Or, alternatively, are closures that lead to $E(k) \sim k^{-\frac{5}{3}}$ to be regarded as wrong?

Although forms like (1.2) are currently receiving a great deal of attention, the

questions just posed have been very largely ignored. Clearly there is a need for extensive discussion of these points. However, we shall not attempt to pursue this here. In the present work we shall take the pragmatic view that the predictions of LET should be compared with experimental results and with the predictions of other closures. Insofar as the $k^{-\frac{5}{3}}$ law is borne out by experimental measurements at high Reynolds numbers, we shall also make comparisons with the Kolmogorov spectrum.

Finally, we would emphasize the special status of the direct-interaction approximation (Kraichnan 1959). This arises not just from its pioneering role but also from the fact it has been the most intensively studied of theories and has had many successes in terms of quantitative predictions (for a summary see Kraichnan 1982). This is true of both its Eulerian (Kraichnan 1959) and Lagrangian (Kraichnan 1965) forms. From our present point of view, we are particularly interested in the ability of the Eulerian DIA to predict the evolution of the energy spectrum in time from arbitrary initial conditions at low-to-moderate Reynolds numbers (Kraichnan 1964). We shall discuss this in more detail later on: for the present we merely note that, under these circumstances, the quantitative predictions of DIA are generally very good but there is some tendency to underestimate energy transfers at high wavenumbers. This shows up particularly in the calculation of the skewness factor (Herring & Kraichnan 1972).

The performance of the Lagrangian-history form of DIA, and its extensions (Kraichnan & Herring 1978; Herring & Kraichnan 1979), will be discussed at a later stage when we deal with the calculation of LET at high Reynolds numbers.

2. The basic equations

In this section we summarize the LET equations for the correlation and propagator functions. We begin by fixing the notation. Let us consider an incompressible fluid occupying a cubical box of side L. At a later stage we shall take the limit $L\to\infty$ (which is required for rigorous isotropy) and summations will then be replaced by integrals. If we let the velocity field be $U_\alpha(\boldsymbol{x},t)$ then the Fourier components of this are defined by

$$U_\alpha(\boldsymbol{x},t)=\sum_{\boldsymbol{k}} U_\alpha(\boldsymbol{k},t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{2.1}$$

The equation of motion may be written as

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)U_\alpha(\boldsymbol{k},t)=\sum_{\boldsymbol{j}} M_{\alpha\beta\gamma}(\boldsymbol{k})\,U_\beta(\boldsymbol{j},t)\,U_\gamma(\boldsymbol{k}+\boldsymbol{j},t), \tag{2.2}$$

and the continuity equation becomes

$$k_\alpha\,U_\alpha(\boldsymbol{k},t)=0, \tag{2.3}$$

where the inertial-transfer operator $M_{\alpha\beta\gamma}(\boldsymbol{k})$ is defined by

$$M_{\alpha\beta\gamma}(\boldsymbol{k})=(2\mathrm{i})^{-1}\{k_\beta D_{\alpha\gamma}(\boldsymbol{k})+k_\gamma D_{\alpha\beta}(\boldsymbol{k})\} \tag{2.4}$$

and

$$D_{\alpha\beta}(\boldsymbol{k})=\delta_{\alpha\beta}-k_\alpha k_\beta|\boldsymbol{k}|^{-2}. \tag{2.5}$$

The pair correlation of velocities may be defined thus:

$$\left(\frac{L}{2\pi}\right)^3\langle U_\alpha(\boldsymbol{k},t)\,U_\beta(-\boldsymbol{k},t')\rangle=Q_{\alpha\beta}(\boldsymbol{k};t,t'), \tag{2.6}$$

where $\langle\ \rangle$ means average value. For isotropic turbulence the correlation function $Q(k;t,t')$ may be introduced through the relationship

$$Q_{\alpha\beta}(\boldsymbol{k};t,t') = D_{\alpha\beta}(\boldsymbol{k})\,Q(k;t,t'), \tag{2.7}$$

and the expression for the energy spectrum follows in the usual way:

$$E(k,t) = 4\pi k^2 Q(k;t,t). \tag{2.8}$$

In LET (see I), the equations for both Q and the exact velocity propagator H are obtained directly from the primitive perturbation series (cf. Wyld 1961; Lee 1965). Renormalization is anticipated by introducing the exact propagator formally thus:

$$U_\alpha(\boldsymbol{k},t) = H_{\alpha\sigma}(\boldsymbol{k};t,s)\,U_\sigma(\boldsymbol{k},s), \tag{2.9}$$

where
$$H_{\alpha\sigma}(\boldsymbol{k};t,s)\,H_{\sigma\beta}(\boldsymbol{k};s,t') = H_{\alpha\beta}(\boldsymbol{k};t,t'), \quad H_{\alpha\sigma}(\boldsymbol{k};t,t) = 1. \tag{2.10}$$

The exact propagator is taken to be statistically sharp. For isotropic turbulence it may be written in the form

$$H_{\alpha\beta}(\boldsymbol{k};t,t') = D_{\alpha\beta}(\boldsymbol{k})\,H(k;t,t'), \tag{2.11}$$

where $H(k;t,t')$ may be referred to as the propagator function.

The derivation of the LET equations for H and Q may be found in I. Here we quote the result for isotropic non-stationary turbulence as follows:

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)H(k;t,t') = W(k;t,t') \quad (t'<t), \tag{2.12}$$

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)Q(k;t,t') = P(k;t,t'), \tag{2.13}$$

and for the diagonal values of the correlation function

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q(k;t,t) = 2P(k;t,t). \tag{2.14}$$

The inertial-transfer terms W and P are given by

$$W(k;t,t') = \int \mathrm{d}^3\boldsymbol{j}\,L_{kju}\left[\int_0^{t'} \frac{\mathrm{d}s H(k;t',s)\,Q(j;t,s)\,Q(|\boldsymbol{k}+\boldsymbol{j}|;t,s)}{Q(k;s,s)} - \int_0^t \mathrm{d}s\,H(k;s,t')\,H(j;t,s)\,Q(|\boldsymbol{k}+\boldsymbol{j}|;t,s)\right] \tag{2.15}$$

and

$$P(k;t,t') = \int \mathrm{d}^3\boldsymbol{j}\,L_{kj\mu}\left[\int_0^{t'} \mathrm{d}s\,H(k;t',s)\,Q(j;t,s)\,Q(|\boldsymbol{k}+\boldsymbol{j}|;t,s) - \int_0^t \mathrm{d}s\,Q(k;s,t')\,H(j;t,s)\,Q(|\boldsymbol{k}+\boldsymbol{j}|;t,s)\right] \tag{2.16}$$

where

$$L_{kj\mu} = \frac{[\mu(k^2+j^2)-kj(1+2\mu^2)]\,(1-\mu^2)\,kj}{k^2+j^2-2\mu kj}, \tag{2.17}$$

and μ is the cosine of the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{j}$.

It should be noted that (2.13), (2.14) and (2.16) are identical (notational differences aside) with corresponding DIA equations, whereas (2.12) and (2.15) differ from their DIA equivalents by the presence of the first term (i.e. the one containing Q^{-1}) on the right-hand side of (2.15) for $W(k;t,t')$.

3. Evolution from arbitrary initial conditions

In formulating the problem of isotropic turbulence, one specifies the statistical ensemble by one's choice of the arbitrary, random, stirring forces. For reasons of convenience these are assumed to have Gaussian statistics; to do work on the system at a constant rate; and to excite predominantly low-k modes of the system. Thus, under the action of the inertial transfer of energy to higher modes, the system should reach a stationary state with some universal behaviour in the higher-k modes. Of course, even if such a system were physically realizable, this formulation reflects the artificiality of the concept, compared with flows of engineering significance (in general, inhomogeneous shear flows).

Nevertheless, the simplest 'real' turbulence problem is generally agreed to be a version of the above formulation; that is, free decay of isotropic turbulence. In practice this is realized in the laboratory by placing a grid in a potential-core flow in a wind (or water) tunnel. Some grid-pitches downstream, the wakes coalesce to form turbulence which is nearly isotropic. The subsequent decay of this turbulence with distance from the grid may be transformed to a time decay in a frame moving with the freestream velocity. In this situation, the concept of stationarity is replaced by self-preservation, where the energy spectrum (suitably scaled) becomes independent of time. We can also expect universal behaviour at high wavenumbers where spectra (again suitably scaled) exhibit similarity. Here the low-k behaviour depends on the arbitrarily chosen initial spectra $E(k, 0)$. (That is, the arbitrariness of the stirring forces at all times is replaced by the arbitrarily prescribed initial conditions.)

Four different initial spectra were used in the calculations for low-to-moderate Reynolds numbers. They are shown in figure 1. Three of them – spectra I, II and III – are identical with spectra B, C and D of Kraichnan (1964). They all peak at the same wavenumber $k_{\max} = 4 \times 2^{\frac{1}{4}} = 4.75683$ (in arbitrary units), and represent successively less-peaked initial distributions of energy. Noteworthy points are that spectrum II is self-preserving under purely viscous decay and spectrum III becomes rapidly self-preserving under the combined actions of inertial transfer and viscous decay.

Spectrum IV is a modified form of spectrum III. It was chosen in the light of comments made by Van Atta & Chen (1969), who compared their measured dissipation spectrum with spectra calculated by Kraichnan (1964). They found that the calculated spectra exhibited maxima at higher wavenumbers than the measured spectra. They suggested that this might be due to the choice of initial spectra, whose maxima were also shifted towards high wavenumbers. Accordingly, spectrum IV has been chosen to test the effect of a lower value of the peak wavenumber, in this case $k_{\max} = 2.0$.

These four trial spectra may each be written in the form

$$E(k, 0) = c_1 k^{c_2} \exp(-c_3 k^{c_4}), \tag{3.1}$$

where the values of the constants for each spectrum are given in table 1, and each spectral form satisfies

$$\int_0^\infty E(k, 0)\, \mathrm{d}k = \tfrac{3}{2}. \tag{3.2}$$

(That is, the mean-square level of the turbulence at $t = 0$ is taken to be unity.)

From the energy spectrum, we can obtain the mean energy $E(t)$ per unit mass, the

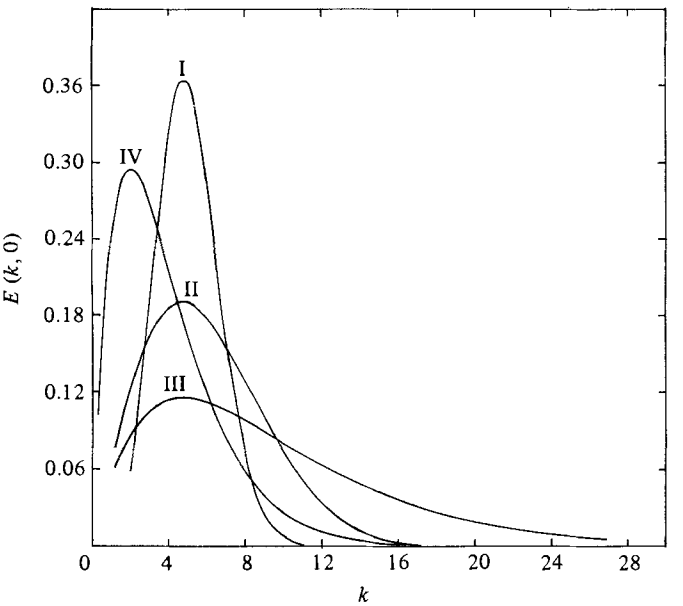

FIGURE 1. Initial wavenumber spectra.

Spectrum number	c_1	c_2	c_3	c_4
I	0.524169×10^{-2}	4	0.883882×10^{-1}	2
II	0.662912×10^{-1}	1	0.220971×10^{-1}	2
III	0.662912×10^{-1}	1	0.210224	1
IV	0.4	1	0.5	1

TABLE 1. Values of constants c in $E(k,0) = c_1 k^{c_2} \exp(-c_3 k^{c_4})$

r.m.s. value of any velocity component $u(t)$ and the rate of dissipation of energy $\epsilon(t)$ per unit mass thus:

$$E(t) = \int_0^\infty E(k,t)\,\mathrm{d}k = \tfrac{3}{2}[u(t)]^2 \tag{3.3}$$

and

$$\epsilon(t) = 2\nu \int_0^\infty k^2 E(k,t)\,\mathrm{d}t. \tag{3.4}$$

It is also convenient to have forms for the transfer spectrum $T(k,t)$:

$$T(k,t) = 8\pi k^2 P(k;t,t), \tag{3.5}$$

and, following Kraichnan (1964), the modal time-correlation $R(k;t,t')$:

$$R(k;t,t') = \frac{Q(k;t,t')}{[Q(k;t,t)\,Q(k;t',t')]^{\frac{1}{2}}} \tag{3.6}$$

such that

$$R(k;t,t) = 1. \tag{3.7}$$

The integral scale $L(t)$ and the Taylor microscale $\lambda(t)$ may be defined by (Batchelor 1971)

$$L(t) = \left[\frac{3\pi}{4}\int_0^\infty k^{-1}E(k,t)\,\mathrm{d}k\right]\Big/E(t), \tag{3.8}$$

$$\lambda(t) = \left[5E(t)\Big/\int_0^\infty k^2E(k,t)\,\mathrm{d}k\right]^{\frac{1}{2}}, \tag{3.9}$$

with associated Reynolds numbers

$$R_L(t) = L(t)\frac{u(t)}{\nu} \tag{3.10}$$

and

$$R_\lambda(t) = \lambda(t)\frac{u(t)}{\nu}. \tag{3.11}$$

As we shall see, the skewness of the longitudinal velocity derivative is perhaps the most sensitive of the integral parameters. It is given by (Batchelor 1971)

$$S(t) = -\frac{\langle(\partial u_1(\boldsymbol{x},t)/\partial x_1)^3\rangle}{\langle(\partial u_1(\boldsymbol{x},t)/\partial x_1)^2\rangle^{\frac{3}{2}}} = \frac{2}{35}\left[\frac{\lambda(t)}{u(t)}\right]^3\int_0^\infty k^2T(k,t)\,\mathrm{d}k. \tag{3.12}$$

We shall normally refer to $S(t)$ just as the skewness factor.

As experimentalists measure only the one-dimensional energy spectrum $\phi_1(k,t)$ we should note the relationship between it and the three-dimensional spectrum. It is given by (Batchelor 1971)

$$\phi_1(k,t) = \frac{1}{2}\int_k^\infty\left(1-\frac{k^2}{p^2}\right)p^{-1}E(p,t)\,\mathrm{d}p. \tag{3.13}$$

Finally, we shall follow Kraichnan (1964) in defining a characteristic wavenumber and velocity

$$k_\mathrm{d} = (15R_\lambda)^{\frac{1}{3}}\lambda^{-1}, \quad v_\mathrm{d} = (R_\lambda/15^{\frac{1}{2}})^{-\frac{1}{3}}u(t). \tag{3.14}$$

Kraichnan found that for low R_λ scaling with k_d and v_d, rather than with the Kolmogorov similarity variables k_s and v_s (for the relationship between k_d and k_s, and v_d and v_s see Kraichnan 1964), gave a better collapse of the data for purposes of studying similarity and self-preservation. In order to facilitate comparisons, we have also used (3.14) in the present work.

4. The numerical analysis

Our objective here is to integrate (2.15)–(2.17) forward in time from arbitrary initial states, viz the initial spectra shown in figure 1. Before discussing the numerical methods we should make two general points.

First, there is the question of how to compare the results of our calculations with results obtained from experiments. As Herring & Kraichnan (1972) have pointed out, approximations of the kind presented here are based on a prescribed initial state with zero triple correlations and the integrations are carried out for relatively short times. In contrast, laboratory experiments involve initial states which are rather complicated and have much longer evolution times. To some extent we have already noted this when we chose spectrum IV, and of course the computer experiments of Orszag & Patterson (1972) go some way to meet this difficulty. We shall discuss these aspects further at a later stage but here we shall deal with a particular point concerning the LET theory.

Referring to (2.11) for the propagator $H(k;t,t')$, the right-hand side is given by (2.15). The first term on the right-hand side is what distinguishes LET from DIA. The particular point we are concerned with is this: the integration over s in the numerator has a convolution character, whereas $Q(k;s,s)$ in the denominator is a single-time quantity. If a stationary state is attained, then the denominator just becomes $Q(k)$, provided the non-stationary state is some time in the remote past so that its effects may be neglected (see I). However, the problem here is that this condition is not satisfied. We have only very short evolution times to a state which is self-preserving (i.e. scaled spectra independent of time). As $Q(k;s,s)$ does not contain either t or t', it is relatively slow in evolving. This can lead to spurious effects during the initial period of evolution (it should be emphasized that this is a matter that only affects the initial period and, even then only when the prescribed initial spectrum is very peaked like, for example, spectrum I). For this reason we have simplified the computational problem by means of the following practical expedient. We have made the approximation that $Q(k;s,s)$ in the denominator in (2.15) is updated to $Q(k;t,t)$. It should be noted that this is not a Markovian approximation; memory effects are still fully represented in the numerator of this term. The approximation was tested by computing both forms for $E(k,0)$ as given by spectrum III, which is self-preserving. There is little difference between the two. Even the evolved skewness factor $S(t)$ differed by less than 1 % between the two cases.

Our second general point is that we used exactly the same numerical methods as used by Kraichnan (1964, 1966), but there was one non-trivial difference in the formulation to which they were applied. In calculating the wavenumber integrals in DIA, Kraichnan worked with the scalar magnitudes of three wavevectors $\boldsymbol{k}, \boldsymbol{j}$ and $\boldsymbol{l}$ (say), which add up to form a vector triangle. In our case we work with $\boldsymbol{k}$ and $\boldsymbol{j}$, and then specify $\boldsymbol{l}$ in terms of $\boldsymbol{k}, \boldsymbol{j}$ and the cosine of the angle between $\boldsymbol{k}$ and $\boldsymbol{j}$, which we call μ. Mathematically, of course, the two procedures are equivalent. The differences between them come in the way they affect rounding-off and truncation errors in the numerical calculations. For this reason we have made our own calculations of DIA (which merely means setting the first term of the right-hand side of (2.15) to zero). This allows us to: (*a*) make a valid comparison of LET with DIA; and (*b*) check our calculations by comparing our results for DIA with those of Kraichnan (1964).

4.1. *Reduction to discrete wavenumbers*

We begin by considering the discrete representation of the wavenumbers k and j. Let the range of these variables be truncated to the finite range $(k_{\text{bot}}, k_{\text{top}})$, where k_{bot} may or may not be zero. We divide this range into N (in general, non-uniform) intervals Δk_n, with n increasing with wavenumber. In each interval we choose a centre value k_n (calculated as the geometric or arithmetic mean of the limits of the interval, depending on whether we employed constant logarithmic or linear steps). The scalar variable μ lies in the range $(+1, -1)$ and we divide this into M constant linear intervals of size $\Delta\mu_m$.

As an example, we may replace (2.15) and (2.16) for the inertial-transfer terms W and P by

$$W_n(t,t') = \sum_{p=1}^{N}\sum_{m=1}^{M} \mathscr{L}_{npm}\left[\int_0^{t'} \mathrm{d}s \frac{H_n(t',s)\,Q_p(t,s)\,Q_{npm}(t,s)}{Q_n(t,t)} - \int_{t'}^{t} \mathrm{d}s\, H_n(s,t')\,H_p(t,s)\,Q_{npm}(t,s)\right], \quad (4.1)$$

$$P_n(t,t') = \sum_{p=1}^{N}\sum_{m=1}^{M} \mathscr{L}_{npm}\left[\int_0^{t'} \mathrm{d}s\, H_n(t',s)\, Q_p(t,s)\, Q_{npm}(t,s) - \int_0^{t} \mathrm{d}s\, Q_n(s,t')\, H_p(t,s)\, Q_{npm}(t,s)\right], \quad (4.2)$$

where $$Q_n(t,t') = Q(k_n;\, t,t'), \quad H_n(t,t') = H(k_n;\, t,t') \quad \text{etc.},$$

with $$\mathscr{L}_{npm} = 2\pi k_p^2\, L_{npm}\, \Delta k_p\, \Delta\mu_m, \quad (4.3)$$

and $$L_{npm} = \frac{[\mu_m(k_n^2+k_p^2) - k_n\, k_p(1+2\mu_m^2)]\, k_n\, k_p(1-\mu_m^2)}{(k_n^2+k_p^2-2k_n\, k_p\, \mu_m)}. \quad (4.4)$$

In (4.1) and (4.2), since $Q(|\boldsymbol{k}+\boldsymbol{j}|;\, t,t')$ is a function of all three scalar variables k, j and μ, we have given it the triple subscript npm when representing it in discrete form.

At this point we should note the advantages of using the present formulation, which leads to a single scalar coefficient $\mathscr{L}_{npm}$ in (4.1) and (4.2), instead of A_{nml} and B_{nml}, which occur in (11.3) and (11.4) of Kraichnan (1964). First, the estimation of $\mathscr{L}_{npm}$, as given by (4.3), is more straightforward than that of A_{nml} and B_{nml}. Secondly, according to Kraichnan (1964), it is necessary to apply a correcting factor to A_{nml} and B_{nml} to eliminate numerical errors when logarithmic steps are used and, say, conditions like $k_n \gg k_p$ are encountered. This entailed additional calculations which, because of our rectangular integration field and the linear discretization of μ, were not found to be necessary in our present formulation with $\mathscr{L}_{npm}$.

4.2. *Reduction to discrete times*

In order to reduce (4.1) and (4.2) to discrete times, we divide the period from $t = 0$ up to the final instant $t = t_f$ into intervals Δt_i ($i = 1, 2, \ldots, I$), which in general will be non-uniform. Here Δt_i is taken to be of the order of the smaller of the characteristic time for convection and the viscous decay time. These are respectively $[u(t)\, k_{\mathrm{top}}]^{-1}$ and $[\nu k^2{}_{\mathrm{top}}]^{-1}$. Let t_i be the time at the end of the ith interval ($t_I = t_f$; $t_0 = 0$). We begin by using the trapezoidal rule to express the integrations over s in (4.1) and (4.2) as sums for $W_n(t_i,t_j)$ and $P_n(t_i,t_j)$. In order to reduce (2.12) to a difference equation, we first approximate $W_n(t_i,t_j)$ by the value

$$W_n(t_i,t_j) = \tfrac{1}{2}[W_n(t_i,t_j) + W_n(t_{i-1},t_j)],$$

which is constant over the interval Δt_i. Integration over the interval Δt_i then yields for (2.12)

$$H_n(t_i,t_j) = \exp(-\nu_n\, \Delta t_i)\, H_n(t_{i-1},t_j) + \frac{1}{2\nu_n}[1-\exp(-\nu_n\, \Delta t_i)]\,[W_n(t_i,t_j) + W_n(t_{i-1},t_j)] \quad (j < i), \quad (4.5)$$

with the initial value $$H_n(t_i,t_i) = 1. \quad (4.6)$$

In a similar fashion we obtain for (2.13)

$$Q_n(t_i,t_j) = \exp(-\nu_n\, \Delta t_i)\, Q_n(t_{i-1},t_j) + \frac{1}{2\nu_n}[1-\exp(-\nu_n\, \Delta t_i)]\,[P_n(t_i,t_j) + P_n(t_{i-1},t_j)], \quad (4.7)$$

with the initial value for this equation given by

$$Q_n(t_0,t_0) = Q_n(0,0) = \frac{E_n(0)}{4\pi k_n^2}, \quad (4.8)$$

where $E_n(0)$ is any specified initial energy spectrum.

To use the implicit integration scheme (4.5) and (4.7) we go through the following stages.

(i) At the ith step, estimate Δt_i as the smaller of the characteristic times for convection and viscous dissipation.

(ii) Next compute temporary values $H'_n(t_i, t_j)$, $Q'_n(t_i, t_j)$ $(j < i)$ and $Q'_n(t_i, t_i)$ by replacing $W_n(t_i, t_j)$, $P_n(t_i, t_j)$ and $P_n(t_i, t_i)$ on the right-hand side by $W_n(t_{i-1}, t_j)$, $P_n(t_{i-1}, t_j)$ and $P_n(t_{i-1}, t_{i-1})$.

(iii) To obtain the permanent values, we then evaluate $W_n(t_i, t_j)$, $P_n(t_i, t_j)$ and $P_n(t_i, t_i)$ by substituting the temporary values for each factor H and Q that appears on the right-hand side with argument i. In the process we use the symmetry of $Q_n(t_i, t_j)$ under interchange of t_i and t_j.

(iv) To start step $i+1$, these W_n and P_n are recomputed using permanent H- and Q-values.

All four stages are then carried out for each value of the index n.

Some of the calculations used the above scheme as written. That is, one temporary and one permanent value were obtained for H_n and Q_n. We shall refer to such calculations as being for *single iteration*. However, one may iterate the predictor–corrector scheme to any order, until the difference between temporary and current values is less than some specified error. We shall refer to such calculations as being for *unlimited iterations*.

5. Results for low-to-moderate Reynolds numbers

Spectra I–III were computed to give evolved microscale Reynolds numbers R_λ of order 20. Spectrum IV was computed to give an evolved $R_\lambda \sim 40$. The various numerical values are summarized in table 2 for all runs, and the corresponding initial values of the integral parameters are given in table 3.

In practice we found that the computation of LET behaved much like that of DIA, but was slower to converge at higher wavenumbers. To give an example, let us consider the computation of spectrum I at the seventh time step, where $t = 0.48L(0)/u(0)$. If we define the temporary value of the spectrum function to be $Q^{(1)}$ and the permanent value to be $Q^{(2)}$, then at $k_n = 2$ LET gives $Q^{(1)} = 0.1159 \times 10^{-2}$ and $Q^{(2)} = 0.1159 \times 10^{-2}$. However, at $k_n = 26.91$ LET gives $Q^{(1)} = 0.1433 \times 10^{-7}$ and $Q^{(2)} = 0.1992 \times 10^{-7}$.

For this reason, we have considered two cases. First, we computed LET for spectra I–III using single iteration and made a detailed comparison with DIA. Secondly, we computed LET for spectra I, III and IV using unlimited iterations, until temporary and permanent values agreed (even at large wavenumbers) within 10%, and made a detailed comparison with DIA and with the results of experiments. In both cases DIA was only calculated for single iteration: our experience bore out Kraichnan's (1964) conclusion that further iteration of the predictor–corrector scheme made no difference to the results for DIA.

5.1. *Comparison of LET with DIA: single iteration and unlimited iterations*

We calculated values of the energy spectrum, the dissipation spectrum, the transfer spectrum and the integral parameters for both LET and DIA for the initial conditions given by spectra I–III. The general qualitative behaviour of both approximations was found to be quite similar, although there were small quantitative differences which we shall discuss in §5.2.

As might be expected from the numerical examples given at the beginning of this

Spectrum number	k_{bot}	k_{top}	Δk†	μ_{bot}	μ_{top}	$\Delta\mu$	μ	Δt_1	t_f‡
I	1.83401	29.3441	$\frac{1}{4}$	+1	−1	−0.1250	0.01189	0.034	0.84
II	1.09051	29.3441	$\frac{1}{4}$	+1	−1	−0.1053	0.01189	0.034	0.95
III	1.09051	41.4989	$\frac{1}{4}$	+1	−1	−0.0952	0.01	0.024	0.65
IV	0.2806	35.9182	$\frac{1}{3}$	+1	−1	−0.0952	0.008	0.027	1.00

† Δk is defined by $k_{n+1}/k_n = 2^{\Delta k}$.

‡ By this time the change in $E(t)$ between any two successive time steps would be anywhere between 7 % and 2 %, depending on the initial peakedness of the spectrum.

TABLE 2. Numerical data for runs at low-to-moderate R_λ

Spectrum number	$E(0)$	$\epsilon(0)$	$u(0)$	$L(0)$	$\lambda(0)$	$R_L(0)$	$R_\lambda(0)$
I	1.4847	1.0092	0.9949	0.5140	0.4182	43.01	34.99
II	1.4631	1.6157	0.9876	0.5218	0.3281	43.34	27.26
III	1.4659	3.9764	0.9886	0.4035	0.1920	39.89	18.98
IV	1.5893	0.6157	1.0293	1.0333	0.4544	132.96	58.47

TABLE 3. Initial values of the integral parameters for low-to-moderate R_λ

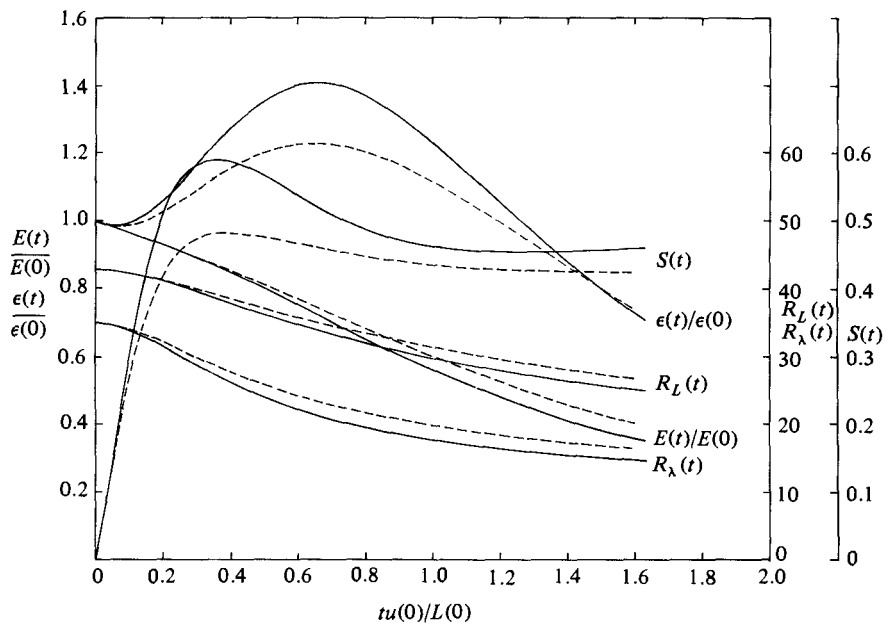

FIGURE 2. Variation of integral parameters; spectrum I: ——, LET (single iteration); – – –, DIA.

section, a detailed comparison revealed no significant difference between LET values of $E(k,t)$, $\nu k^2 E(k,t)$ and $T(k,t)$ calculated using single iteration and unlimited iterations. Even the integral parameters remain quite unchanged by further iterations of LET. However, there is one notable exception. A comparison of figures 2 and 3 shows that the LET value for the evolved skewness is increased by about 7 % following further iteration. In contrast, the result for DIA is found to change by less than 1 %.

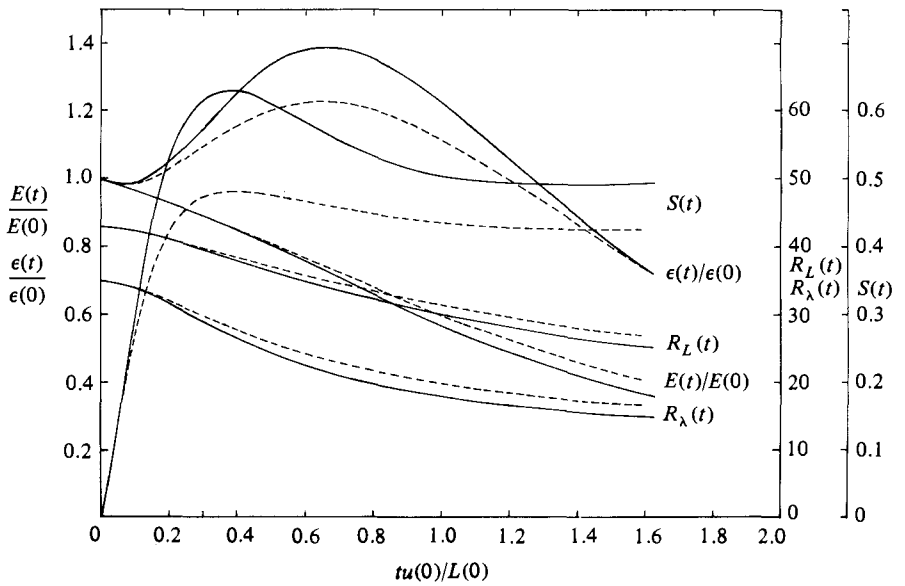

FIGURE 3. Variation of integral parameters; spectrum I: ——, LET (unlimited iterations); – – –, DIA.

Overall, similar results were found when we compared the LET results for spectrum III from only single iteration with those from unlimited iterations. That is, increasing the number of iterations made little difference to anything except the skewness factor.

All of this may seem to be labouring the point. But in §6 we shall present calculations of LET at large Reynolds numbers. This was a formidably large calculation and, in order to reduce machine time to tolerable levels, it was necessary to restrict most of the calculation to single iteration. It was therefore important to establish, as clearly as possible, the effect of this restriction.

5.2. *General calculation of LET: unlimited iterations*

In figures 4–8 we present results for initial conditions given by spectrum I for both LET and DIA. From figure 4 we see that the LET energy spectrum evolves slightly faster than that of DIA. This was also found to be the case with Spectra II and III. This may be attributed to LET being more efficient than DIA at transferring energy to higher wavenumbers. This is shown up clearly by the higher tails of the LET curves for $k^2E(k,t)$ and $T(k,t)$ in figures 5 and 6.

One qualitative difference between the two approximations was the development of kinks in the evolving LET energy spectrum but not in the corresponding DIA spectrum. (These kinks show up more clearly in the dissipation spectrum and can also be seen in the transfer spectrum.) This behaviour did not occur with either spectrum II or spectrum III. Presumably it was due to the more efficient energy transfer mechanism of LET failing to cope with an initial spectrum (i.e. spectrum I) which was highly peaked. The development of kinks may well not be an artifact of LET. Similar kinks have been found experimentally by Stewart & Townsend (1951).

The variation of integral parameters already shown in figure 3 tends to bear out the above points about the comparison of LET and DIA. In particular we should note the dissipation rate $\epsilon(t)$ and the skewness $S(t)$ are both larger for LET than for DIA. In fact the value of $S(t)$ is some 16% larger than that from DIA. In view of

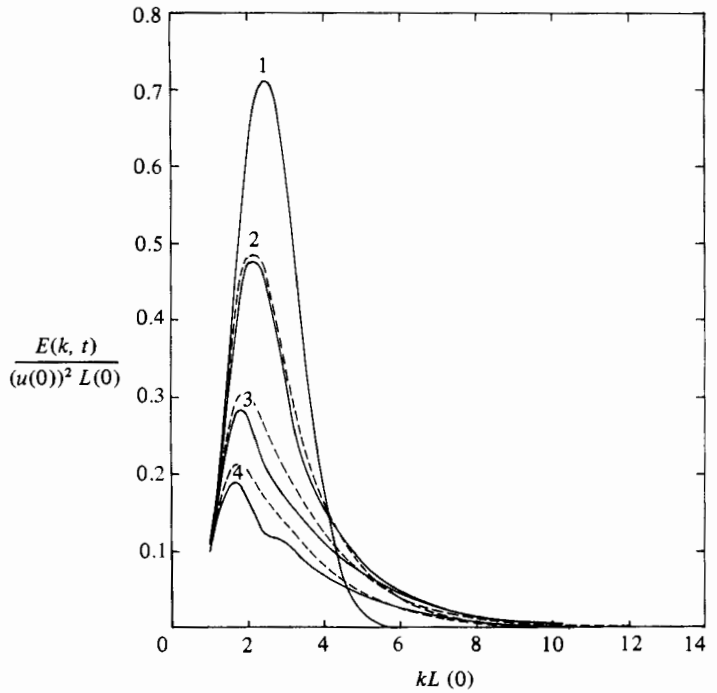

FIGURE 4. Evolution of energy spectrum; spectrum I: ——, LET; – – –, DIA. 1, $tu(0)/L(0) = 0$; 2, $tu(0)/L(0) = 0.5$; 3, $tu(0)/L(0) = 1.0$; 4, $tu(0)/L(0) = 1.6$.

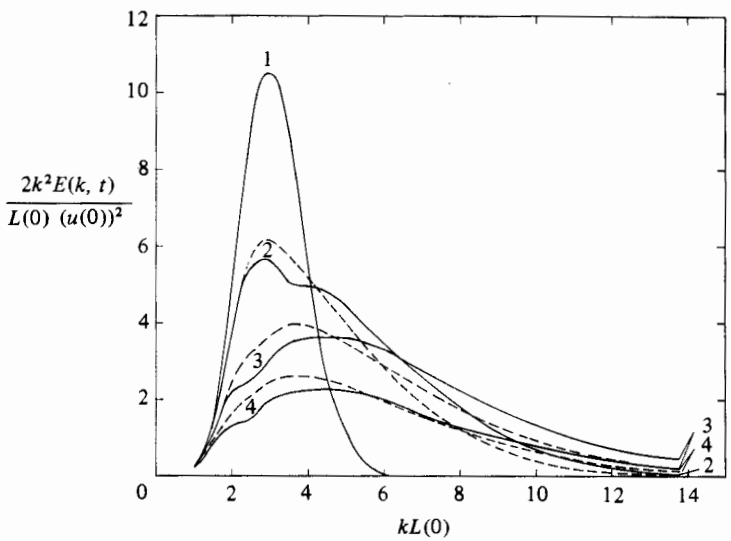

FIGURE 5. Evolution of dissipation spectrum; spectrum I: ——, LET; – – – –, DIA. 1, $tu(0)/L(0) = 0$; 2, $tu(0)/L(0) = 0.5$; 3, $tu(0)/L(0) = 1.0$; 4, $tu(0)/L(0) = 1.6$.

the compatibility of LET with the Kolmogorov distribution, this is an interesting result. When comparing DIA with the test field model, Herring & Kraichnan (1972) surmised that the underestimation of $S(t)$ by DIA was a real physical effect, associated with lack of random convection invariance.

The differences between LET and DIA may also be expected to show up in the way they handle two-time correlations. Plots of the modal time correlation $R(k, t_1 - t)$

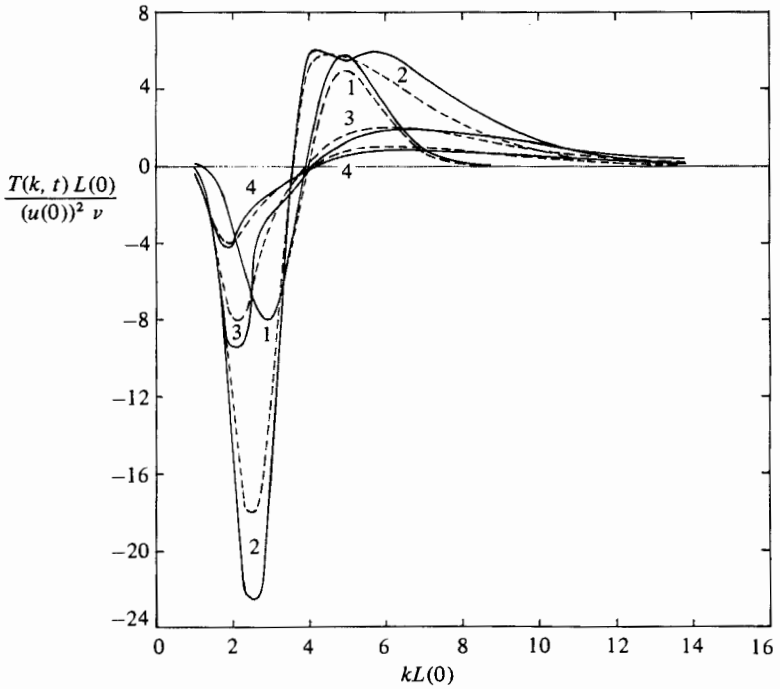

FIGURE 6. Evolution of transfer spectrum; spectrum I: ——, LET; – – –, DIA. 1, $tu(0)/L(0) = 0.1$; 2, $tu(0)/L(0) = 0.5$; 3, $tu(0)/L(0) = 1.0$; 4, $tu(0)/L(0) = 1.6$.

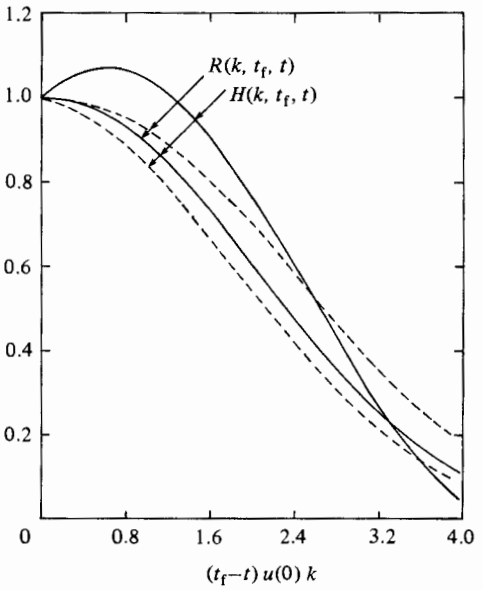

FIGURE 7. Variation of modal time-correlation and response functions at a relatively low wavenumber; spectrum I; $kL(0) = 2.4$, $t_f\,u(0)/L(0) = 1.6$: ——, LET; – – –, DIA.

and the propagator function $H(k, t_f - t)$ for LET and DIA (strictly, the second quantity for DIA is the response function $G(k, t_f - t)$ in Kraichnan's notation) are shown in figures 7 and 8. In general terms, all four curves behave in much the same way. But, apart from the obvious quantitative differences there is one important qualitative difference. In LET H is generally larger than R, while the reverse is true

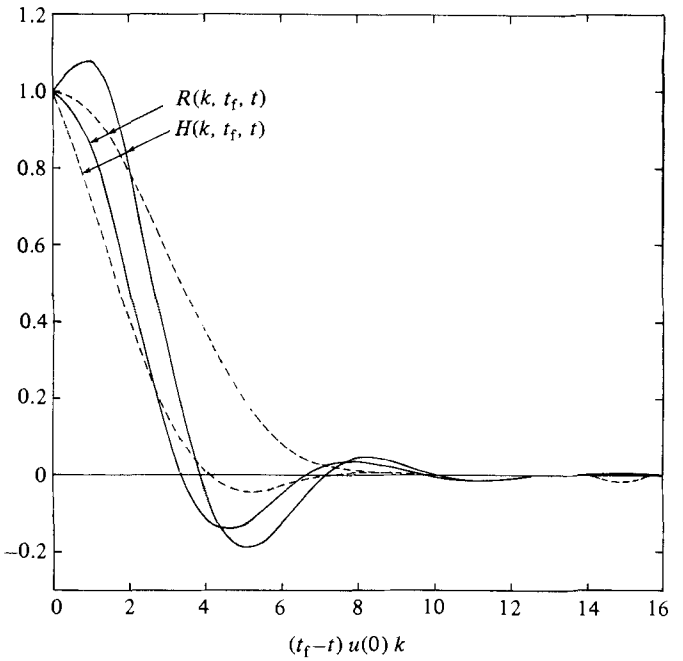

FIGURE 8. Variation of modal time-correlation and response functions at a relatively high wavenumber; spectrum I; $kL(0) = 9.8$, $t_f\, u(0)/L(0) = 1.6$: ——, LET; – – –, DIA.

of DIA. This difference is less apparent at the higher wavenumber (see figure 8), where at intermediate times the LET values for H and R tend to group rather closely with the DIA value for H. Also, while all four curves exhibit unphysical behaviour in the form of negative values at long times, the LET value of H also exceeds unity at short times.

In figures 9 and 10 we show the effect of initial spectrum shape on the evolved energy spectrum and the evolved dissipation spectrum (in the latter case we have used Kraichnan's DIA similarity variables as defined by equation (3.14) of the present paper). The development of similarity at higher wavenumbers is quite apparent. We have only shown results for LET here, but the general behaviour is much like that of DIA (Kraichnan 1964), although (consistent with the above discussion) the LET case appears to be evolving slightly faster.

The development of self-preservation for spectrum I and spectrum III may be seen from figures 11 and 12. Here, the scaled energy spectrum is plotted against wavenumber scaled by $\lambda(t)$. Clearly there is some persistence of initial shape, but, apart from that, one may conclude that (as in the case of DIA) there is considerable self-preservation for both initial spectra. Differences between Spectrum I and Spectrum III show up more clearly in the evolved dissipation spectrum (not presented here), but these are small. In particular, a plot of the dissipation spectrum indicates more-or-less perfect self-preservation for spectrum III (initially self-preserving).

5.3. *Comparison with experiment*

Figure 13 shows comparisons of the one-dimensional dissipation spectrum for both LET and DIA with some experimental results. Clearly both approximations agree quite well with experiment. The calculations are presented for $R_\lambda \sim 38$ (LET) and $R_\lambda \sim 41$ (DIA). Here the comparison is with experimental results at $R_\lambda \sim 39$ (Stewart

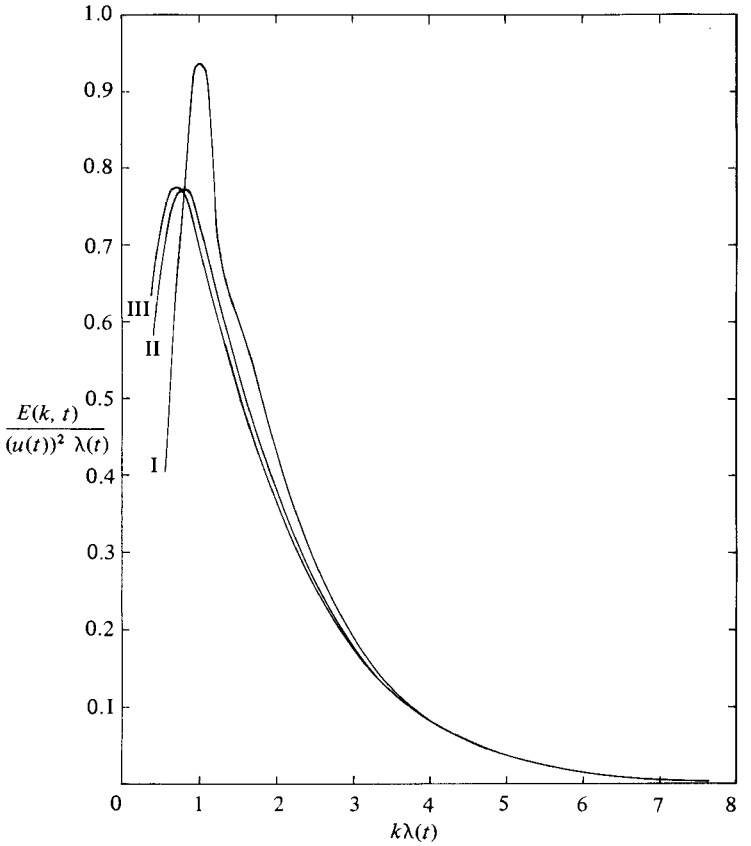

FIGURE 9. Effect of initial spectrum-shape on the evolved LET energy spectrum:

Spectrum	$tu(0)/L(0)$	R_λ
I	1.2	16.3
II	1.6	16.6
III	1.9	16.6

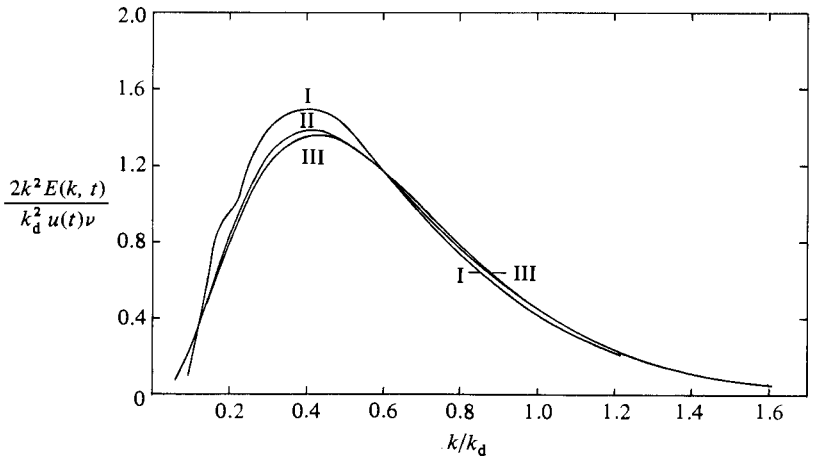

FIGURE 10. Effect of initial spectrum-shape on the evolved LET dissipation spectrum:

Spectrum	$tu(0)/L(0)$	R_λ
I	1.2	16.3
II	1.6	16.6
III	1.9	16.6

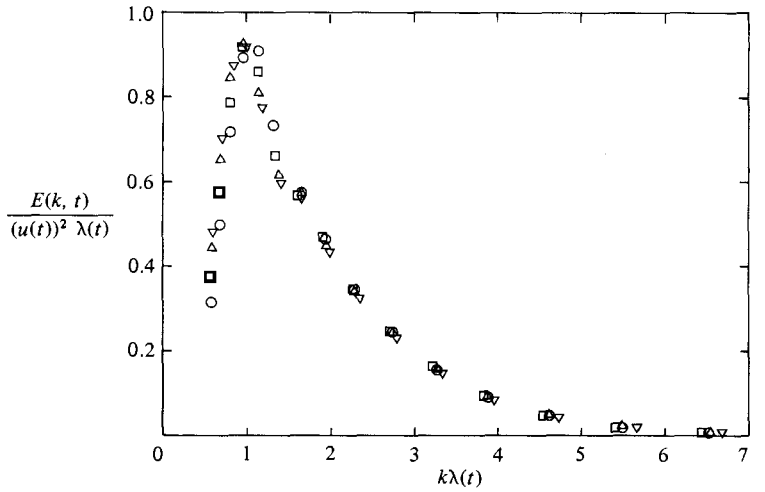

FIGURE 11. Evolution of LET energy spectrum; spectrum I:

	$tu(0)/L(0)$	R_λ
○	0.9	19.1
□	1.1	17.1
△	1.4	15.6
▽	1.6	14.8

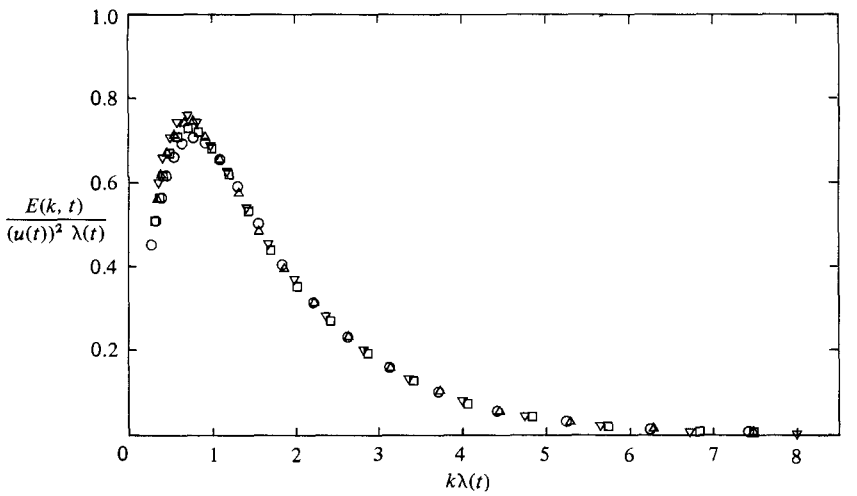

FIGURE 12. Evolution of LET energy spectrum; spectrum III:

	$tu(0)/L(0)$	R_λ
○	0.6	17.5
□	0.9	17.1
△	1.3	16.9
▽	1.6	16.7

& Townsend 1951), $R_\lambda \sim 49$ and ~ 35 (Chen 1968), $R_\lambda \sim 38$ and ~ 37 (Comte-Bellot & Corrsin 1971), and $R_\lambda \sim 45$ (Frenkiel & Klebanoff 1971).

From figure 13 the differences shown between LET and DIA really are rather small. A better quantitative test comes from a comparison of their predictions for the skewness. As Herring & Kraichnan (1972) have pointed out in their comparison of various theories, curves for the skewness as a function of time seem to be the most sensitive measure of the differences between theories. In figure 14 we plot the results

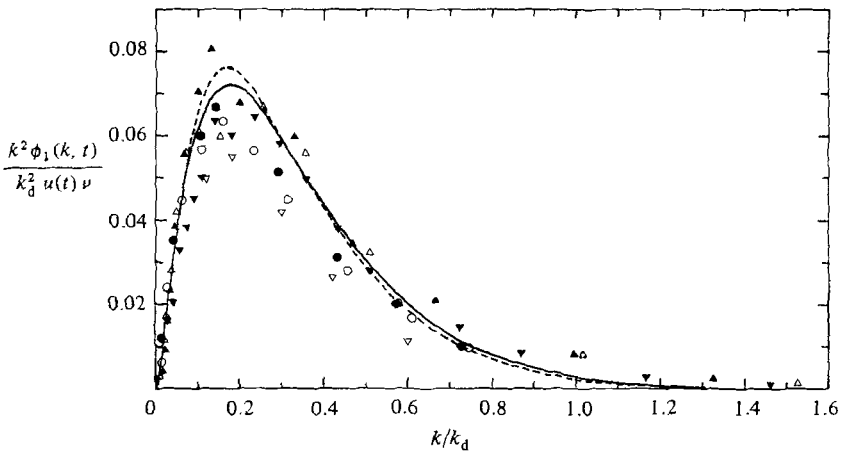

FIGURE 13. Comparison of evolved one-dimensional dissipation spectra at moderate values of R_λ. Computed results for spectrum IV at $tu(0)/L(0) = 1.0$: ——, LET, $R_\lambda = 38.2$; – – –, DIA, $R_\lambda = 40.5$. Experimental results: ▽, $R_\lambda = 39.4$, $x/M = 30$ (Stewart & Townsend 1951); ○, $R_\lambda = 49.0$ and, ●, $R_\lambda = 35.0$ at $x/M = 48$ (Chen 1968); △, $R_\lambda = 38.1$, $x/M = 240$ and, ▲, $R_\lambda = 36.6$, $x/M = 385$ (Comte-Bellot & Corrsin 1971); ▼, $R_\lambda = 45.2$, $x/M = 174.4$ (Frenkiel & Klebanoff 1971).

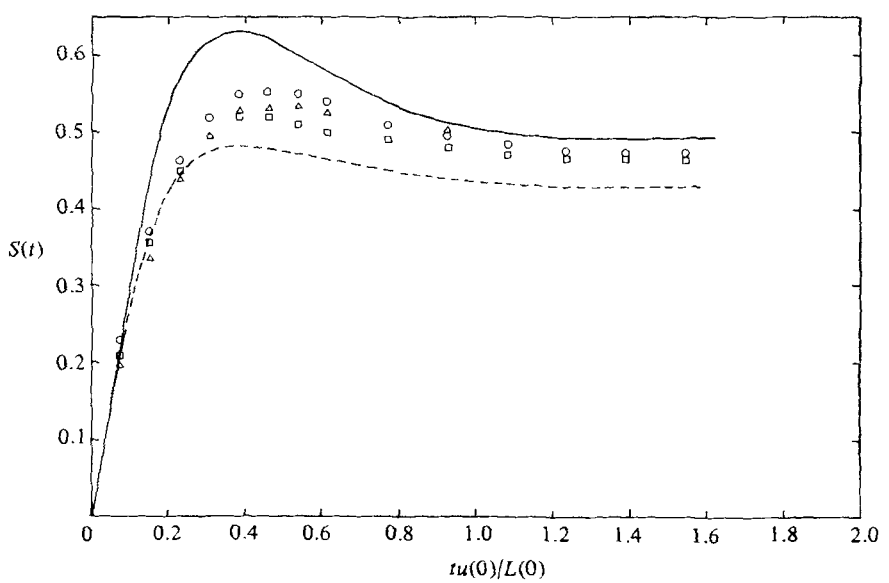

FIGURE 14. Evolution of skewness factor; spectrum I: ——, LET; – – –, DIA; ○, □, △, direct numerical simulations (Orszag & Patterson 1972).

for $S(t)$ from LET and DIA, and compare them with the results of the computer simulation by Orszag & Patterson (1972). In both the computer simulation and the present calculations, the initial conditions were given by spectrum I. Clearly the LET results agrees quite well with that of the computer simulation: certainly as well as the test field model (Herring & Kraichnan 1972). Also it should be noted that our calculation for DIA gives $S(t)$ about 4% larger than in that reference.

We shall return to comparisons of the skewness factor with experiment at a later stage when we discuss the calculations for large Reynolds numbers. To conclude this section we shall make a few remarks about two-time correlations.

The differences between the LET and DIA forms of the modal time correlation have already been noted. The only source of data for further comparison seems to be the computer experiment of Orszag & Patterson (1972). These authors note that the DIA modal time correlation $R(k;t,t_1)$ agrees quite well with their result obtained from their numerical simulation, but that at high wavenumbers there is serious disagreement (of the order of 20 %). We have made our own comparison, which confirms this behaviour for DIA, but suggests that the LET modal time correlation agrees much more closely with the simulation. Again, this result seems to be consistent with the higher energy transfer rate, dissipation rate and evolved skewness factor of the LET closure.

5.4. *Discussion*

The interpretation of our results depends of course on the accuracy of the numerical procedures. As stated earlier, we used the same numerical methods as Kraichnan (1964), but applied them to a slightly different mathematical formulation. We were able to cross-check our calculations by comparing our results for DIA with those of Kraichnan (1964). In all cases we found good general agreement. In the particular case of the skewness, our calculation of the DIA value for $S(t)$ was less than 5 % larger than Kraichnan's.

Independent calculations of LET and DIA have been carried out by Cliffe (1983 private communications) on the CRAY-1 computer at AERE, Harwell. Cliffe has used Kraichnan's (1964) method adapted only to allow it to be run on a vector processor. Broadly, his results confirm ours for the systematic differences between LET and DIA, but it is interesting to note that his calculation of the evolved skewness is 5.5 % below ours.

A measure of the overall accuracy of the numerical calculation can be obtained from an examination of the energy-balance equation. From (2.14), and using (2.8) and (3.5), we obtain the well-known form

$$\frac{\partial E(k,t)}{\partial t}+2\nu k^2 E(k,t) = T(k,t).$$

Integrating with respect to k yields the energy balance

$$\frac{\mathrm{d}E}{\mathrm{d}t}+\epsilon = 0,$$

and the overall error Δ may be written as

$$\Delta = \frac{1}{\epsilon}\frac{\mathrm{d}E}{\mathrm{d}t}+1.$$

Evaluating Δ from our data for the various runs, we obtain estimates for Δ as follows:

spectrum I: $\Delta \sim 10^{-2}$ at small t, $\Delta \sim 10^{-3}$ at large t;
spectrum II: $\Delta \sim 10^{-2}$ at small t, $\Delta \sim 10^{-4}$ at large t;
spectrum III: $\Delta \sim 10^{-2}$ at small t, $\Delta \sim 10^{-4}$ at large t;
spectrum IV: $\Delta \sim 10^{-3}$ at small t, $\Delta \sim 10^{-4}$ at large t.

These values of Δ are much the same order of magnitude as the results reported by Herring & Kraichnan (1972). In all, therefore, our calculations seem to achieve much the same level of accuracy as others reported in the literature.

When the LET theory was reported in I, its main claim was that, although in general form much like DIA, it had (unlike DIA) the Kolmogorov distribution as its

infinite-Reynolds-number solution. (N.B. All remarks here refer to Eulerian formulations.) However, there was at least the possibility that compatibility with the Kolmogorov spectrum has been achieved at the expense of low-Reynolds-number behaviour. In short, LET might not be able to emulate DIA in predicting decaying isotropic turbulence.

These doubts would seem to be effectively banished by the present work. Clearly our results show that LET is not inferior to DIA in predicting the various parameters of grid turbulence. Moreover, LET exhibits higher rates of energy transfer and higher values of the evolved skewness $S(t)$ than DIA. Presumably this may be connected with the ability of LET to give the $k^{-\frac{5}{3}}$ law as the solution at infinite Reynolds number. In all, the behaviour of LET R_λ up to ~ 40 is quite encouraging. In §6 we shall present the results of calculations for LET at large but finite values of R_λ.

6. Results for large Reynolds numbers

The equations were integrated forward in time from an initial spectrum shape prescribed by spectrum V: it was given by

$$\text{spectrum V:} \quad E(k,0) = 2\pi k^{-\frac{5}{3}}. \tag{6.1}$$

This spectrum was chosen, along with a value of the kinematic viscosity $\nu = 0.008$, to permit a close comparison with the computations of the Lagrangian-history closures ALHDI and SBALDHI (Herring & Kraichnan 1979). The initial data for the run are summarized as follows:

energy/unit mass	$E(0) = 40.256$;
dissipation rate	$\epsilon(0) = 22.507$;
r.m.s. velocity	$u(0) = 5.180$;
Taylor microscale	$\lambda(0) = 0.378$;
integral scale	$L(0) = 0.853$;
microscale Reynolds number	$R_\lambda(0) = 244.95$;
integral-scale Reynolds number	$R_l(0) = 5524$.

(For definitions of these parameters see §3.)

For the calculations presented here $k_{\text{bot}} = 0.1114$, $k_{\text{top}} = 71.8601$ and $\Delta k = \frac{1}{3}$ octave. Also for the integral over μ, we took $M = 28$ and $\Delta\mu = -0.07143$. The time step Δt was taken to be of the order of the smaller of the two characteristic times $t_c = [u(t)\, k_{\text{top}}]^{-1}$ and $t_\nu = [\nu k_{\text{top}}^2]^{-1}$. In the present calculation, the values of these times at the beginning of the first time step were $t_c = 0.0027$ and $t_\nu = 0.0242$. We took the time step to be $\Delta t = 2t_c$ throughout the calculation.

The calculations reported here were quite costly in terms of machine time. If we take one of the calculations at low R_λ to provide a standard of comparison, then the case of spectrum I took $\sim 2\frac{1}{2}$ h on the ICL 2972 to reach $t_f = 0.84$ in twenty time steps. Typical machine terms were ~ 35 s to go from t_1 to t_2 and $\sim 1.1 \times 10^3$ s to go from t_{18} to t_{19}.

On the same machine, the present calculation at high R_λ took ~ 260 h to reach $t_f = 0.3$ in 55 time steps. The time required for the second time step was $\sim 2.6 \times 10^2$ s, and for the last time step $\sim 4.6 \times 10^4$ s! Thus we only present a very limited set of calculations (although they are in fact quite adequate for a comparison with both Lagrangian-history theories and with experimental results).

The results reported here are for $\Delta k = \frac{1}{3}$ octave, corresponding to 28 logarithmic intervals in the given range of wavenumbers. Tests were carried out with a finer mesh

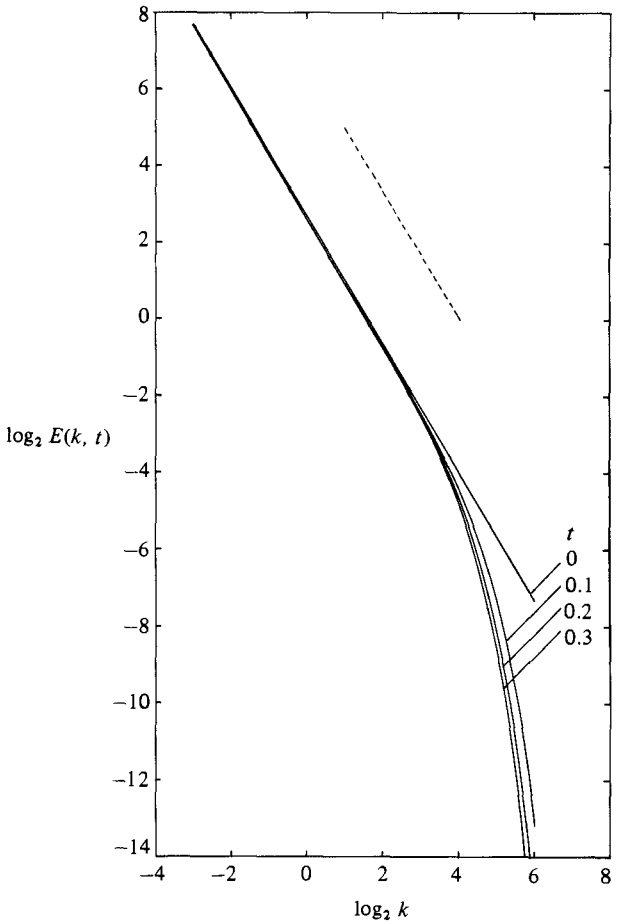

FIGURE 15. Evolution of three-dimensional energy spectrum; spectrum V.

of $\Delta k = 0.1333$ octave (corresponding to 70 wavenumber intervals) but these were abandoned at the eighth time step as the machine time required had become much too large. A comparison of results for the two different meshes showed that the skewness $S(t)$ was changed by less than 2% at the eighth time step.

A comparison was also made of the difference between single iteration and unlimited iterations of the predictor–corrector scheme (see §§4 and 5.1) up to the 13th time step. Again the skewness (the most sensitive of the integral parameters) varied by less than 2% between the two cases at the thirteenth time step. Thus, in the interests of keeping the machine time to reasonable levels, single iteration was employed thereafter.

6.1. *Evolution of the LET equations with time*

The evolution of the energy spectrum from the initial form given by spectrum V is shown in figure 15, where we plot $\log_2 E(k,t)$ against $\log_2 k$. This form of graph shows little change in the inertial range as time goes on, but the development of the roll-off at higher wavenumbers due to viscous dissipation is really quite marked.

In figure 16 the evolution of the various integral parameters is shown. Two aspects of this graph are worth a particular mention here. First, it may be seen that the microscale Reynolds number R_λ evolved from an initial value of ~ 245 to a final value

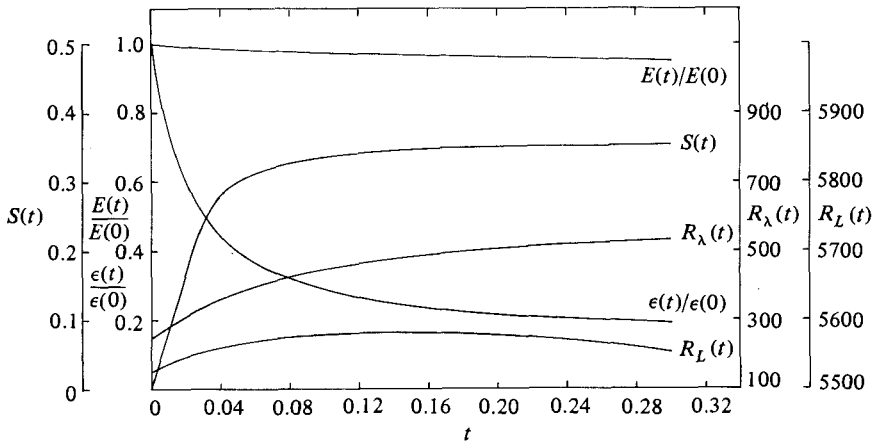

FIGURE 16. Variation of integral parameters; spectrum V.

of ~ 533. Secondly, the skewness reached a plateau with an asymptotic value of $S(t)$ ~ 0.35. This is somewhat lower than the value found (~ 0.5) for the smaller values of R_λ. It is also (as we shall see) very much lower than the values obtained at large R_λ for the Lagrangian-history theories (Herring & Kraichnan 1979).

6.2. *Comparison with experiment and with ALHDI and SBALHDI*

The comparisons with experimental results and with the predictions of ALHDI and SBALHDI (Herring & Kraichnan 1979) are shown in figures 17–19. Of these, figures 17 and 18 show the evolved one-dimensional LET energy spectrum compared respectively with representative experimental results and with the two Lagrangian-history theories. Here the scaling in the terms of the Kolmogorov wavenumber k_s given by

$$k_s = (\epsilon/\nu^3)^{\frac{1}{4}}. \tag{6.2}$$

In both cases, the agreement is good. However, clearly (although there is little in it), the agreement between LET and SBALDHI in the inertial range is particularly close.

The Kolmogorov constant α may also be obtained from the calculations. The LET value for this turns out to be ~ 2.3, compared with $\alpha \sim 2.1$ for SBALHDI and $\alpha \sim 1.8$ for ALHDI. The first two values may seem on the high side but it is worth noting that Pao's (1965) analysis of the experimental data shows that values of α up to 2.2 (and possibly larger) are not incompatible with measured spectra.

The skewness $S(t)$ has previously been found to be the most sensitive indicator when distinguishing between theories (Herring & Kraichnan 1972). Values of the skewness for LET, ALHDI and SBALHDI are shown as functions of time in figure 19. It is interesting to note that the difference between the two Lagrangian-history theories is about the same as that between SBALHDI and LET. We also note that both Lagrangian-history theories seem (more than LET) to be still evolving at $t = 0.3$.

Finally, in figure 20 we present both theoretical and experimental values of evolved skewness factor over a wide range of Reynolds numbers. Clearly the scatter of the experimental data is so large that none of the theoretical values may be said to be incompatible with experiment. However, it is tempting to suggest that the distribution (and general trend with R_λ) of the experimental results tends to favour the low value of $S(t)$ predicted by LET at large Reynolds numbers.

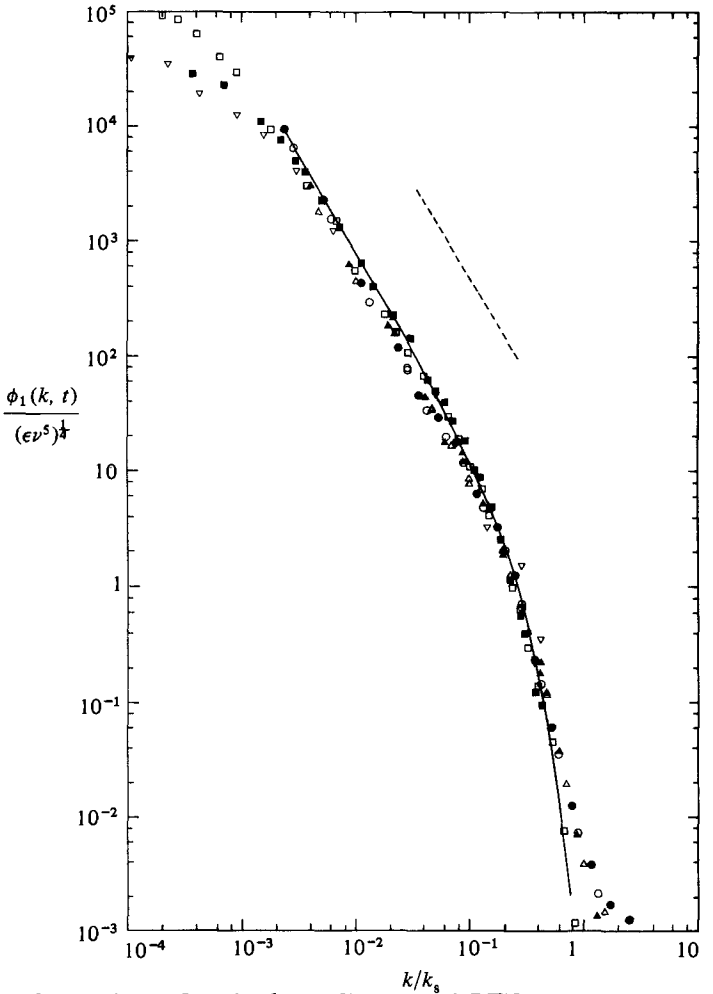

FIGURE 17. Comparison of evolved one-dimensional LET energy spectrum with experimental results. Theory: ——, LET, $R_\lambda \sim 533$. Experiments: ○, ●, △, ▲, $R_\lambda \sim 2000$ (2/2/60, Grant *et al.* 1962); ■, $R_\lambda \sim 538$ (Kistler & Vrebalovich 1966); ▽, $R_\lambda \sim 308$ (Uberoi & Freymuth 1969); □, $R_\lambda \sim 850$ (Coantic & Favre 1974). – – –, $k^{-\frac{5}{3}}$.

6.3. *Comparison of LET with ALHDI and SBALHDI at low R_λ*

In §5 we saw that LET behaved very much like DIA at low values of R_λ. In this part of the paper we have found that LET behaved very much like SBALHDI at high Reynolds numbers (with the exception of the evolved skewness $S(t)$). Herring & Kraichnan (1979) have calculated ALHDI and SBALHDI at low Reynolds numbers using as initial conditions spectrum I and spectrum III. Logically, therefore, we should complete our present comparisons with the Lagrangian-history theories by examining the case of low Reynolds numbers.

In figure 21 we show the three-dimensional dissipation spectrum as predicted by LET and DIA, ALHDI and SBALHDI (Herring & Kraichnan 1979). At smaller wavenumbers it is clear that LET agrees closely with the two Lagrangian-history theories. At higher wavenumbers DIA, SBALHDI and LET are in close agreement. However, if we take the overall behaviour of the four theories at all wavenumbers, then clearly LET and SBALHDI agree much more closely than any other pair of theories.

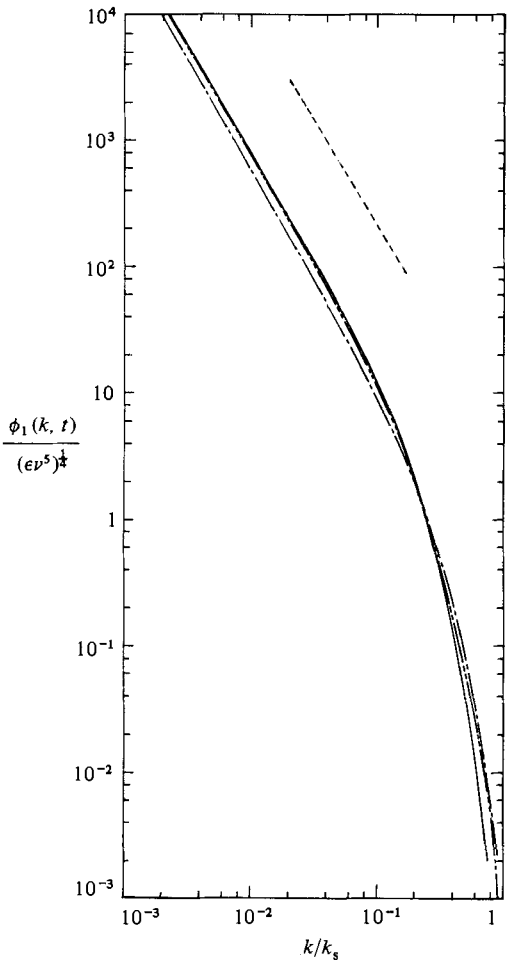

FIGURE 18. Comparison of evolved one-dimensional LET energy spectrum with other theoretical results: ——, LET; — — —, ALHDI; — · — · —, SBALHDI (Herring & Kraichnan 1979: values from their figure 9 at $t = 0.3$ replotted for this comparison); – – –, $k^{-\frac{5}{3}}$.

7. Conclusion

From the results presented in the present paper it seems reasonable to claim that the LET theory offers good quantitative predictions of decaying isotropic turbulence over a wide range of Reynolds numbers. Furthermore, it seems that the purely Eulerian LET theory behaves very much like the Lagrangian-history theory SBA-LHDI at both high and low values of the Reynolds number. Indeed, with the exception of the skewness $S(t)$, the two theories probably agree well within any relative error due to the differences between our numerical calculations and those of Herring & Kraichnan (1979). From our point of view this is an encouraging result. We feel that it justifies our submitting LET to further tests, such as passive scalar convection by isotropic turbulence.

This will be the subject of further work, but we shall conclude this paper with a few general remarks on turbulence closure approximations. In particular, we restrict our attention to renormalized perturbation theories (of which LET is one: a list is given in §1).

It could be argued that the strength of RPT approaches lies in their generality and

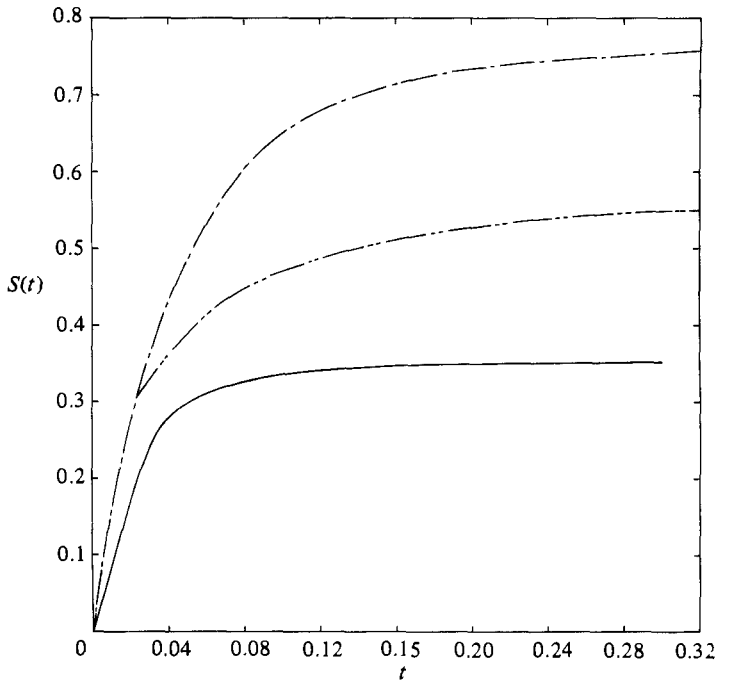

FIGURE 19. Evolution of skewness factor; comparison with other theories: ——, LET; — — —, ALHDI; — — — —, SBALHDI (Herring & Kraichnan 1979).

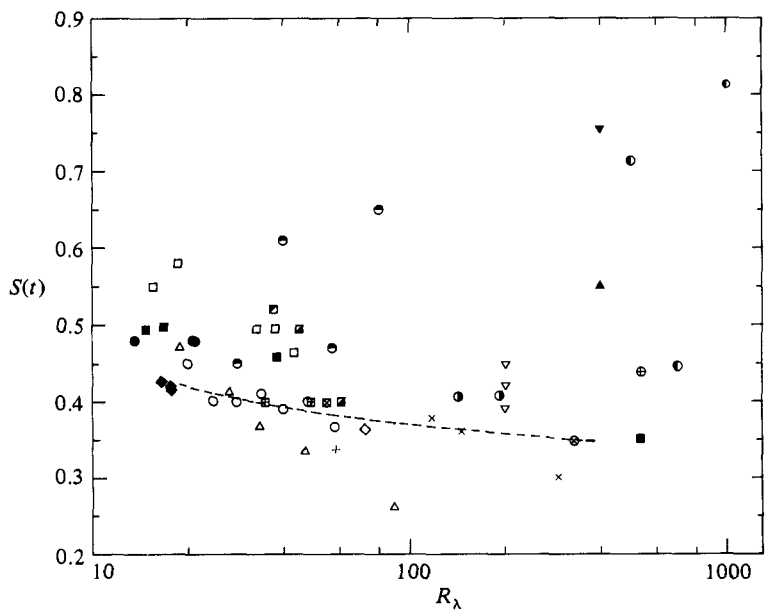

FIGURE 20. Comparison of evolved skewness factor. Computed results: ■, LET; ◆, DIA; ●, direct numerical simulation (Orszag & Patterson 1972); ▼, ALHDI and, ▲, SBALHDI (Herring & Kraichnan 1979); ◓, direct numerical simulation (Brachet *et al.* 1983). Experimental results: ○, Batchelor & Townsend (1949)*; △, Stewart & Townsend (1951); ×, Comte-Bellot (1965)*; ⊠, Frenkiel & Klebanoff (1967); ⊞, Van Atta & Chen (1969); ▽, Wyngaard & Tennekes (1970); ◐, Gibson *et al.* (1970)*; ◇, Kuo & Corrsin (1971); ◩, Frenkiel & Klebanoff (1971); ⊕, Friehe *et al.* (1972)*; ⊗, Betchov & Lorenzen (1974)*; +, Elena *et al.* (1977)*; ◑, Tavoularis (1978)*; □, Bennett & Corrsin (1978); — — —, Ueda & Hinze (1975). (* as cited in Tavoularis, Bennett & Corrsin (1978).)

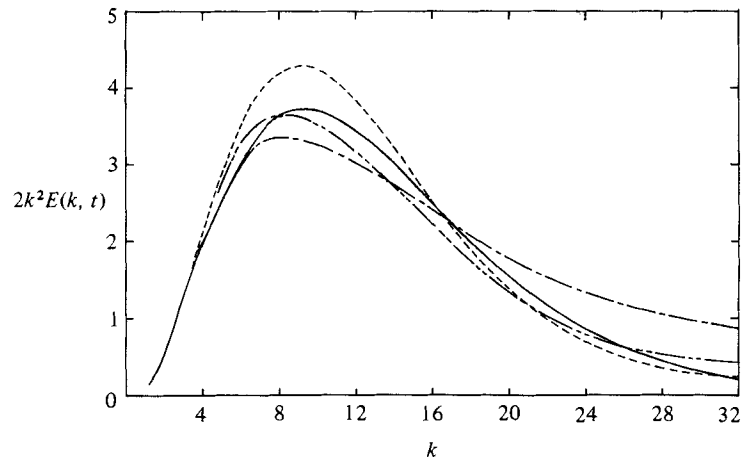

FIGURE 21. Comparison of evolved three-dimensional dissipation spectrum at a low value of R_λ with other theoretical results; initial spectrum as given by Spectrum III: ——, LET; – – –, DIA, — – —, ALHDI; — — — —, SBALHDI (Herring & Kraichnan 1979); $t = 0.6$.

in the absence of *ad hoc* assumptions or disposable constants (particular theories might make more specific claims, e.g. the model representation which guarantees realizability for DIA: Kraichnan 1959). Yet one would not need to be unduly cynical to argue that RPTs (without exception) are cut off from fundamental status by their inability to predict their own errors, and from engineering utility by their enormous complexity when formulated for inhomogeneous turbulence.

There seems to be a growing belief (e.g. Nelkin 1974, 1975; Forster, Nelson & Sulem 1977) that the answer to the first of these problems lies with the renormalization group, which has had its successes with critical phenomena (Wilson 1975). At this stage it can, of course, be no more than a belief; but it is one that we share (McComb 1982; McComb & Shanmugasundaram 1983). At the same time, we agree with Kraichnan (1982) that RPTs have had many successes and should not be underestimated.

The second problem – the analytical complexity of RPTs in engineering flows – is really one of dimensionality. Without the simplifications of homogeneity and isotropy, the calculations in this paper would simply be too large for present-day computers. What is needed is an attack on the problem of analytical reduction of the second-order equations for shear flows, in order to reduce their complexity to the order of (say) the equations computed in the present work. Little appears to have been done on this topic (e.g. Edwards & McComb 1972; Leslie 1973). The reason for this is probably lack of confidence in existing RPTs. Granted that the first problem of *a priori* accuracy lacks a solution, then one would at least look for consistent successes on a hierarchy of easier problems, leading up to a simple shear flow. Disturbingly, this has been lacking. If we take the simplest non-trivial problem to be the one studied here, i.e. free isotropic decay, and consider the performance of DIA, then the picture has not been encouraging. At low R_λ Eulerian DIA works well, but fails at high R_λ. At high R_λ ALHDI is successful, but less good at low R_λ. There are many other points of this sort that can be made, and a further discussion will be found in Kraichnan & Herring (1978).

In view of this, the performance of SBALHDI is distinctly encouraging. However, our hope would be that if LET can survive further tests (e.g. tubulent diffusion), then its relatively simple structure (compared to the Lagrangian-history theories) would

provide a starting point for simple shear flows, such as the free jet or well-developed pipe flow.

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